The Energy of a Dynamical Wave-Emitting System in General Relativity

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Abstract

The problem of energy and its localization in general relativity is critically reexamined. The Tolman energy integral for the Eddington spinning rod is analyzed in detail and evaluated apart from a single term. It is shown that a higher order iteration is required to find its value. Details of techniques to solve mathematically challenging problems of motion with powerful computing resources are provided. The next phase of following a system from static to dynamic to final quasi-static state is described.

PACS numbers: 04.20.Cv, 04.30.-w

1 Introduction

It has been said that "a good problem shows its worth by fighting back". By that criterion, surely the problem of energy in general relativity is a very good one indeed: researchers continually return to it in spite of the fact that many might regard it as a "solved" problem. An important part of the "problem" aspect derives from the difficult issue of energy localizability in general relativity. Einstein [23], Eddington [9] and others argued that energy is inherently non-localizable in general relativity while others such as Bondi [27] dismissed this notion. Through the years, various authors [17] [22] have addressed the problem and recently, a new group of researchers have entered the debate [18].

While the localizability issue is of some interest even for static or stationary systems, of particular interest and importance is the issue in the case of a non-stationary gravity-wave-emitting system. This is due in part to the fact that although gravity waves have been researched intensively over many years, there is only sparse observational evidence, and that not actually direct, of the very existence of gravity waves. Moreover the existence of gravity waves carrying energy would seemingly constitute a necessary condition for the viability of a quantum theory of gravity. Efforts to detect gravitational waves directly have emerged as a major research activity in recent years.

Various aspects of gravity waves led the first author [2] to the hypothesis that energy in general relativity is localized in the regions of the non-vanishing energy-momentum tensor T_i^k (henceforth the "localization hypothesis"). For example, plane fronted gravitational waves with parallel rays have been shown to be of the Kerr-Schild class and for the latter, it has been shown that all components of the pseudotensor vanish *globally* [15]. This is in stark contrast to the truly tensorial corresponding case of plane electromagnetic waves that demonstrate a physically unambiguous flux of energy, both theoretically and observationally. Since there is a link between the Bondi news function and the pseudotensor [24], the same limitations would be drawn from this route of analysis. Another example is that of Bonnor's [30] matching the aspherical Szekeres dust collapse metrics to a Schwarzschild exterior indicating a lack of energy flux. In this regard, it is also interesting to recall remarks by Pirani [4]: "Suppose, for example that a Schwarzschild particle is disturbed from static spherical symmetry by an internal agency, radiates for some time, and finally is restored to static spherical symmetry. Is its total mass necessarily the same as before?" Very recent support for the localization

hypothesis comes from the work of Bringley [18] who has shown that Bonnor's directed beam of radiation is a Kerr-Schild metric.

Indications contrary to the localization hypothesis come from two directions: firstly, there is the gravitational geon construct, a bundle of gravitational waves that self-gravitate into a spherical ball with an exterior Schwarzschild metric indicating mass yet with a spacetime devoid of any energy-momentum tensor or singularities. However, we demonstrated that the time scale for the evolution of the geon was necessarily of the same order as the period of the constituent waves themselves, in contradiction with the assumptions entering into the construct of the geon itself; a perturbation analysis to test the requirement of quasi-stability results in a contradiction [7]. Secondly, there are the imploding/exploding Brill waves, again claimed to present mass without an energy-momentum tensor. However, this construct proceeds from an instant of time symmetry and it is at least questionable whether such an assumption is compatible with nonlinearly interacting waves. This second source of possible contradiction to the localization hypothesis will have to be considered in great detail. In spite of all the years that have since passed and all the papers since written, the question remains. As well, it is to be noted that the localization of energy in non-gravitational physics is in the region of non-vanishing T^{00}. Hence the localization hypothesis may be viewed as a generalization of established physical phenomena and from this vantage point, not justifiably regarded as controversial.

If the localization hypothesis should prove to be correct, it would have fundamental consequences. First, it would imply that gravity waves in vacuum (assuming that they exist and there are ample reasons to believe that they do) would *not* be carriers of

energy, in conformity with the Kerr-Schild aspect. This notion challenges the very meaning that we give to the word "wave", as a disturbance that *carries energy.* Second, without an energy aspect to the waves, it would be difficult to argue in favour of a quantum theory of gravity. In our earlier papers, we noted that while all particles and fields exist *within* spacetime, gravity in essence *is* spacetime itself, i.e. it is intrinsically different conceptually. Thus, it would not be surprising if the role of energy in general relativity should prove to be fundamentally different from that in other areas of physics.

Tolman [10] had found an expression for the total energy of a stationary system including the contribution from gravity as an integral over the region of the energy-momentum tensor. We had focused on this in earlier work [12] and more recently [6] considered the time-rate of change of the Tolman integral at one instant as a vehicle to test the necessity condition for an energy loss via gravitational waves from the classic Eddington spinning rod [9]. The idea was that in the course of the evolution from initial stationarity to final (at least asymptotic) stationarity at both points at which the Tolman integral registers the energy, the Tolman integral should have undergone a change if energy were really being emitted. While the application of the time rate of change at one particular instant does not test with rigour the necessity condition for an energy loss for reasons that will be discussed later, it could possibly serve as a useful at least partial indicator as to how the gravitational contribution to the energy is being altered. At that time, the indications were that the Tolman integral was not changing. In this paper, we first derive a simpler expression for the time-rate of change of the Tolman integral. We then re-examine the Tolman integral more critically and find that the complete proper analysis is far more complex than originally envisaged.

More information regarding the fields than that provided by Eddington and that we had used previously is required and this is furnished. In the course of the analysis, we focus upon the techniques of integration over the coordinates of the inertial frame and over the co-moving frame, using Dirac delta (a time-honoured helpful calculational tool) and step functions as required to properly delineate the source region and motion. These techniques will hopefully be found to be useful for researchers in the future.

We find that all of the terms of the form field/T_{ik} product cancel and we are left with three time derivatives of a trace quadrupole moment to evaluate, much as in traditional flux calculations which employ the untraced quadrupole expression [11] first derived by Einstein [23]. However, for the latter, the quadrupole terms are squared and hence are only required to low order of accuracy. By contrast, in the present work, the quadrupole trace is not squared for the Tolman integral computation and hence this quantity is required to higher order of accuracy, to a second iteration of the field equations. While this could possibly be examined in future research, it is valuable to assess the broader picture. As Bondi, Bonnor, Feynman and others through the years had noted, the truly reliable approach would be to examine the energy of a system at initial and final stationary states of the system that has an intermediate wave-emitting phase. If there is a change, then the existence and extent of an energy loss can be evaluated without ambiguity. The technique described above only examines the rate of change of the Tolman integral at an instant of the dynamic phase. Actually, this rate should be integrated over the lifetime of the dynamic phase for a complete picture. There is an added motivation to do so when one considers that the traditional sources studied have been assumed to be periodic as in the present problem, and as Papapetrou showed

many years ago, periodic sources are incompatible with the Einstein field equations. Thus, the assumption of periodicity is at best an approximation which one hopes is adequate when one undertakes analysis with the periodic assumption. Indeed if the system has developed over a long period, then questions arise regarding the possibility that non-linearities could develop to lead to a different field than is being envisaged. Moreover, the start-up and wind-down phases of the motion are not being considered and they could have an important contribution. To this end, we are presently in the process of analyzing a system with an explicit relatively short evolution in time from a static to a dynamic to a quasi-static final state. This has the advantage of providing the desirable total history as described and discussed above as well as mitigating any onset of non-linear effects due to the relative brevity of the evolution.

For completeness, we review some of the fundamentals of the energy problem in general relativity and the work of Eddington and Tolman in Sec. 2. In Sec. 3, the manner in which we set out to evaluate the time rate of change of the Tolman integral at a particular instant is developed and the framework for its application to the Eddington spinning rod is discussed. We illustrate the techniques that are employed to take time derivatives, comparing the manner of calculating in both co-moving and inertial frames. In Sec. 4, the details of the coordinate dependence for the Eddington problem are given. We supply the technical details that form the transition from the required time rate of change of the formal Tolman integral to the Maple program that is used to compute it in Sec. 5, and we determine the time rate of change of the Tolman integral up to one final term. In Sec. 6, we provide a summary and concluding discussions regarding the weak points of the assumptions leading to the previous results as well as a discriminating test

and suggested future directions.

2 The Energy Problem

We recall that the generally covariant energy-momentum conservation laws required for general relativity [19]

$$T^k_{i;k} = 0. \tag{1}$$

can be cast into the form of an ordinary vanishing divergence

$$\frac{\partial}{\partial x^k}\left(\sqrt{-g}(T^k_i + t^k_i)\right) = 0 \tag{2}$$

by the introduction of the energy-momentum pseudotensor t^k_i [23]. The pseudotensor carries the specifically gravitational contribution to energy and momentum. It is a complicated expression involving the metric tensor and its first partial derivatives. Thus, global conservation laws can be extended into general relativity but at the expense of the inclusion of this non-tensorial object. As a result, the determination of a *localized* expression for energy-momentum becomes problematic as the pseudotensor, unlike T^k_i, can be made to vanish at any pre-selected point by the proper choice of coordinates.

Einstein considered a weak perturbation of Minkowski space

$$g_{ik} = \eta_{ik} + h_{ik} \tag{3}$$

in conjunction with the field equations

$$G^k_i = \frac{8\pi G}{c^4} T^k_i. \tag{4}$$

With the field perturbation h_{ik}, he computed the pseudotensorial Poynting vector flux to lowest non-vanishing order in v/c from a bounded system emitting gravitational waves

using the right hand side of (5) below as

$$\frac{\partial}{\partial t}\int\left(\sqrt{-g}(T_0^0+t_0^0)\right)dV=-c\oint\left(\sqrt{-g}t_0^\alpha\right)dS_\alpha \tag{5}$$

where the integral on the left hand side was regarded as the energy E, extended throughout space via t_0^0.

An alternative approach is to proceed with a multipolar expansion of the field as in electromagnetism to express the rate of energy loss to lowest order as

$$\dot{E}=-\frac{G}{45c^5}\left(\frac{d^3}{dt^3}\right)D_{\alpha\beta}\left(\frac{d^3}{dt^3}\right)D^{\alpha\beta} \tag{6}$$

where $D_{\alpha\beta}$ is the mass quadrupole tensor [11]. As applied to the Eddington spinning rod, both methods yield the energy loss rate

$$\frac{dE}{dt}=-\frac{32GI^2\omega^6}{5c^5} \tag{7}$$

where I is the moment of inertia, and ω is the angular velocity of the rigid (to lowest order) Eddington rod.

Later, Eddington [9] used a local approach analogous to that of radiation damping calculations in electromagnetism. With the conservation laws (1) expressed as ($\mathcal{T}^{ab}$ is defined as $\sqrt{-g}T^{ab}$)

$$\frac{\partial}{\partial x^k}\left(\mathcal{T}_i^k\right)=\frac{1}{2}\mathcal{T}^{ab}\frac{\partial h_{ab}}{\partial x^i}, \tag{8}$$

he integrated over a region just beyond the confines of the source and using Gauss' theorem, found a useful expression for the time rate of change of the integral of $\mathcal{T}_0^0$ as

$$\frac{\partial}{\partial t}\int\mathcal{T}_0^0 dV=\frac{1}{2}\int\mathcal{T}^{ab}\frac{\partial}{\partial t}h_{ab}dV. \tag{9}$$

The beauty of (9) is that it enables one to focus upon the energy- momentum tensor part of the energy expression without bringing in the non- localized pseudotensorial components as in (5). Indeed , after an interesting sequence of argumentation in his book [9], Eddington concluded that $\mathcal{T}^{00}$ was the appropriate density of energy in general relativity. Rather than the asymptotic field that is required in the flux calculation of (5), the local field is required in calculations using the RHS of (9). Eddington found this in the harmonic gauge as

$$h_{ab} = -4 \int \left[\frac{T'_{ab} - \frac{1}{2}\eta_{ab} T'}{r(1 - \frac{v_r}{c})} \right]_{\rm ret} dV'. \tag{10}$$

To bring the elements of the retarded integral to a common time, Eddington performed a present time expansion as

$$\left[\frac{T'_{ab}}{r\left(1 - \frac{v_r}{c}\right)} \right]_{\rm ret} = \frac{T'_{ab}}{r} - \frac{d}{dt} T'_{ab} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{dt^n} \left(r^{n-1} T'_{ab} \right). \tag{11}$$

He considered the rod spinning in the $x - y$ plane and evaluated the right hand side of (9) with the rod lying along the x axis at $t = 0$. When the three-volume element dV is at x and element dV' is at x', the distance between them at time t is

$$r = \sqrt{x^2 + x'^2 - 2xx' \cos \omega t} \tag{12}$$

which is used in conjunction with (10), (11) and (9) and t is set to zero after the differentiations.

Simplifications occur because if T^{ab} is any non-vanishing component at t $= 0$, T'_{ab} will be an even function of time so odd order derivatives vanish and only the series

$$
\begin{aligned}
\frac{\partial}{\partial t}\left[\frac{T'_{ab}}{r(1-\frac{v_r}{c})}\right] = & -\frac{d^2}{dt^2}T'_{ab} - \frac{1}{6}\frac{d^4}{dt^4}\left(T'_{ab}(x^2 + x'^2 - 2xx'\cos\omega t)\right) \\
& -\frac{1}{120}\frac{d^6}{dt^6}\left(T'_{ab}(x^2 + x'^2 - 2xx'\cos\omega t)^2\right) + \cdots
\end{aligned}
\tag{13}
$$

is required. With the rod spinning in the $x-y$ plane, the required transformation is

$$
\begin{aligned}
\bar{x}^0 &\equiv \bar{t} = t, \\
\bar{x}^1 &\equiv \bar{x} = x\cos\omega t + y\sin\omega t, \\
\bar{x}^3 &\equiv \bar{z} = z \\
\bar{x}^2 &\equiv \bar{y} = -x\sin\omega t + y\cos\omega t
\end{aligned}
\tag{14}
$$

where the $\bar{x}^i$-coordinates are co-moving with the rod and the x^i-coordinates are the inertial coordinates. This transformation is of sufficient accuracy for the calculations that we will require.

We will delve into greater detail than did Eddington. As the desired calculation is a perturbative one, we begin with the expression of the energy-momentum tensor in the co-moving frame to the lowest order,

$$
\begin{aligned}
\bar{T}^{00} &= \sigma\delta(\bar{y}), \\
\bar{T}^{11} &= \sigma\frac{\omega^2}{2}(\bar{x}^2 - a^2)\delta(\bar{y})
\end{aligned}
$$

and all other components are zero. We can transform this into the inertial coordinates

using(14) to get

$$
\begin{aligned}
T^{00} &= \bar{T}^{00} \\
T^{01} &= -\omega y \bar{T}^{00} \\
T^{02} &= \omega x \bar{T}^{00} \\
T^{11} &= -\omega^2 y^2 \bar{T}^{00} + \cos^2 \omega t \, \bar{T}^{11} \\
T^{12} &= -\omega^2 xy \bar{T}^{00} + \cos \omega t \, \sin \omega t \, \bar{T}^{11} \\
T^{22} &= \omega^2 x^2 \bar{T}^{00} + \sin^2 \omega t \, \bar{T}^{11}.
\end{aligned}
\tag{15}
$$

Expressed in co-moving variables, they are

$$
\begin{aligned}
T^{00} &= \bar{T}^{00} \\
T^{01} &= -\omega(\bar{y} \cos \omega t + \bar{x} \sin \omega t) \bar{T}^{00} \\
T^{02} &= \omega(\bar{x} \cos \omega t - \bar{y} \sin \omega t) \bar{T}^{00} \\
T^{11} &= -\omega^2 (\bar{y} \cos \omega t + \bar{x} \sin \omega t)^2 \bar{T}^{00} + \cos^2 \omega t \, \bar{T}^{11} \\
T^{12} &= -\omega^2 (\bar{x} \cos \omega t - \bar{y} \sin \omega t)(\bar{y} \cos \omega t + \bar{x} \sin \omega t) \bar{T}^{00} + \cos \omega t \, \sin \omega t \, \bar{T}^{11} \\
T^{22} &= \omega^2 (\bar{x} \cos \omega t - \bar{y} \sin \omega t)^2 \bar{T}^{00} + \sin^2 \omega t \, \bar{T}^{11}.
\end{aligned}
\tag{16}
$$

Eddington used the information he required from (16) and again he found the result as in (7).

Eddington chose to regard the integral of $\mathcal{T}_0^0$ as the total energy of a system. In the past, we have referred to this integral as the material or "kinetic" energy but this is not a truly adequate description because the root of the determinant of the total metric tensor is included rather than that of the spatial part of the metric tensor that would provide proper three-volume. Thus, some field contribution is included in the Eddington expression. In later years, Tolman [10] (see also [11] for a more elegant

derivation) showed that for stationary systems, the total energy is actually

$$E = \int \left(\mathcal{T}_0^0 - \mathcal{T}_\alpha^\alpha \right) dV. \tag{17}$$

This is particularly attractive in that it is a wholly localized expression for bounded distributions and most importantly, it accounts fully for the contribution to the energy from the gravitational field. Moreover, by comparison with (5), we can identify the pseudotensorial gravitational contribution to the energy as deriving from the integral of the trace of the stresses/momentum flux densities. It was this portion of the gravitational contribution to the energy of a bounded stationary system that Eddington missed, his work pre-dating that of Tolman. We have seen its importance in various ways [12]. While we do not have a localized expression for energy in the non-stationary case, it is nevertheless of interest to ascertain whether or not this Tolman integral varies during a non-stationary phase. This is because it is the Tolman integral that will be the arbiter of energy change when the system eventually returns to stationarity. It must be stressed, however, that the information that we can glean at present is incomplete because we are incapable of measuring the change in the Tolman integral for the complete history stretching from the stationary start to the (asymptotically) stationary end. As well, there is the assumption of periodicity with the dynamic field emerging from the source at a given instant without regard to prior history. These aspects add further uncertainty.

There are alternative expressions for the mass of an isolated system. The Bondi mass [25] m(u) is evaluated on the null cone. Since the transition to stationarity is an asymptotic one, the evaluation of the mass of system that is left behind after the dynamic period would entail the additional condition of allowing t to approach infinity

using the Bondi expression, adding to the complexity.

Another choice is that due to Arnowitt, Deser and Misner (ADM) [20]. Their elegant formula expresses the mass of a stationary system as an asymptotic spacelike surface integral involving the asymptotic metric tensor components. We [21] generalized the use of the ADM integral for non-stationary systems by treating it as a Poynting vector but we now see this as a pseudotensorial flux with the attendant ambiguities that this presents.

In our view, the Tolman integral is particularly attractive in that it is evaluated over the domain of the source and hence it yields the desired final state characteristic when the energy-momentum tensor no longer varies in time. This is because the evaluation of the Tolman mass at that point of the system that remains behind will be the same from that time onwards as the waves continue to infinity. What is being evaluated thereby is the end-state active gravitational mass of the material source including the contributions from its gravitational field and this is precisely what we seek in gauging whether or not there has been an energy loss of the system. One is not burdened by conditions on the asymptotic field in this approach.

3 Formalism to Compute the Variation of the Tolman Integral

By raising an index, we can readily re-express the conservation law (8) as [6]

$$\mathcal{T}^{lk}_{,k} = F^{l} \tag{18}$$

where

$$F^l = \frac{1}{2}\mathcal{T}^{ab}h_{ab,i}g^{il} + g^{il}_{,k}\mathcal{T}^k_i \tag{19}$$

As before, in what follows, we will designate x^i as the approximately inertial Cartesian coordinates. The Gauss theorem in conjunction with (18) gives

$$\frac{\partial}{\partial x^0}\int(\mathcal{T}^{\delta 0}x^\gamma + \mathcal{T}^{\gamma 0}x^\delta)dV = 2\int \mathcal{T}^{\gamma\delta}dV + \int(F^\delta x^\gamma + F^\gamma x^\delta)dV \tag{20}$$

Similarly, we multiply by two $x's$ and integrate to get

$$\frac{\partial}{\partial x^0}\int \mathcal{T}^{00}x^\gamma x^\delta dV = \int(\mathcal{T}^{0\delta}x^\gamma + \mathcal{T}^{0\gamma}x^\delta)dV + \int F^0 x^\gamma x^\delta dV. \tag{21}$$

After taking $\frac{\partial}{\partial x^0}$ of (21), eliminating $\frac{\partial}{\partial x^0}\int(\mathcal{T}^{\gamma 0}x^\delta + \mathcal{T}^{\delta 0}x^\gamma)dV$ by using (20) and setting $\delta = \gamma$ yields

$$\int \mathcal{T}^{\gamma\gamma}dV = \frac{1}{2}\frac{\partial^2}{\partial(x^0)^2}\int \mathcal{T}^{00}x^\gamma x^\gamma dV - \int F^\gamma x^\gamma dV - \frac{1}{2}\frac{\partial}{\partial x^0}\int F^0 x^\gamma x^\gamma dV \tag{22}$$

.

The integral of the spatial trace of the energy momentum tensor is required to complete the Tolman integral and its time rate of change is readily found as

$$\begin{aligned}\frac{\partial}{\partial x^0}\int \mathcal{T}^\gamma_\gamma dV = \frac{\partial}{\partial x^0}\int \mathcal{T}^{\gamma k}g_{k\gamma}dV &= \int T^{\gamma\beta}_{,0}g_{\beta\gamma}dV + \int \mathcal{T}^{\gamma 0}_{,0}g_{0\gamma}dV + \int \mathcal{T}^{\gamma k}g_{k\gamma,0}dV \\ &= -\frac{\partial}{\partial x^0}\int \mathcal{T}^{\gamma\gamma}dV + \int \mathcal{T}^{\gamma\beta}_{,0}h_{\beta\gamma}dV + \int \mathcal{T}^{\gamma 0}_{,0}h_{0\gamma}dV \\ &\quad + \int \mathcal{T}^{\gamma 0}h_{0\gamma,0}dV + \int \mathcal{T}^{\gamma\beta}h_{\beta\gamma,0}dV\end{aligned} \tag{23}$$

using (3) and $\eta_{\alpha\beta}$= diagonal$(-1,-1,-1)$.

The substitution of (22) into (23) yields

$$\begin{aligned}\frac{\partial}{\partial x^0}\int \mathcal{T}^{\gamma}_{\gamma} dV = & -\frac{1}{2}\frac{\partial^3}{\partial (x^0)^3}\int \mathcal{T}^{00} x^\alpha x^\alpha dV + \frac{\partial}{\partial x^0}\int F^\alpha x^\alpha dV \\ & + \frac{1}{2}\frac{\partial^2}{\partial (x^0)^2}\int F^0 x^\alpha x^\alpha dV \\ & + \int \mathcal{T}^{\alpha k}_{,0} h_{\alpha k} dV + \int \mathcal{T}^{\alpha k} h_{k\alpha,0} dV\end{aligned} \tag{24}$$

This is a simpler form of the required expression than we had found in earlier work ([6]). The second to fifth terms on the RHS of (24) are of order h_{ik} times T_{ik} and hence of the familiar form of the RHS of the Eddington equation (9). However the first term on the RHS of (24) must be considered more carefully as there is no metric perturbation in it. While the untraced version of this term is only required to low order in the traditional flux calculations of the past, this is not the case in the present context because we do not square this term to find the time rate of change of the Tolman integral. It is required to higher accuracy and this will require a second iteration of the field equations.

It will also be of interest to evaluate the time rate of change of the "material" angular momentum [29]

$$\frac{dL^{\gamma\delta}}{dt} = \frac{d}{dt}\int (x^\gamma \mathcal{T}^{0\delta} - x^\delta \mathcal{T}^{0\gamma}) dV. \tag{25}$$

Using (18) and (19) and Gauss' theorem, this becomes

$$\frac{dL^{12}}{dt} = \int (F^2 x^1 - F^1 x^2) dV \tag{26}$$

for the z component of angular momentum.

As an illustration of the techniques that we will employ, we compute the time derivatives of the mass quadrupole moment tensor

$$D_{\alpha\beta} = \int \mu (3x^\mu x^\nu \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\beta} x^\mu x^\nu \delta_{\mu\nu})\, dV \tag{27}$$

to the lowest order.

We can perform the calculation in the co-moving frame (as is commonly done with electromagnetic radiation analysis) or in the inertial reference frame. The latter has time-dependent parts arising from the density $\mu = \sigma\delta(z)\delta(\bar{y})(H(a-\bar{x}) + H(a+\bar{x}) - 1)$ where H is the step-function. Equation (27) becomes

$$\begin{aligned} D_{\alpha\beta} &= \int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\int_{-a}^{a} \sigma\delta(z)\delta(\bar{y})(H(a-\bar{x}) + H(a+\bar{x}) - 1)(3x^\alpha x^\beta - \delta_{\alpha\beta}x^\mu x^\nu \delta_{\mu\nu})\, dxdydz \\ &= \int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\int_{-a\cos\omega t}^{a\cos\omega t} \sigma\delta(z)\delta(-x\sin\omega t + y\cos\omega t)(3x^\alpha x^\beta - \delta_{\alpha\beta}x^\mu x^\nu \delta_{\mu\nu})\, dxdydz \end{aligned}$$

for some $\epsilon > 0$, large enough to enclose the Dirac-delta function. The time-dependence of the limits of integration arises by virtue of the confinement of the source as viewed in the inertial frame. The rod is confined by two factors, the Dirac-delta function, $\delta(y\cos\omega t - x\sin\omega t)$, that maintains its position along the co-moving $\bar{x}$ axis and a double-step function, $(H(a-(x\cos\omega t + y\sin\omega t)) + H(a+(x\cos\omega t + y\sin\omega t)) - 1)$, that specifies its length. The double-step function forces the limits of integration to be time-dependent. Had we used fixed limits of integration, $x \in [-a, a]$ as opposed to $x \in [-a\cos\omega t, a\cos\omega t]$, it would have signified that the rod was expanding as it sweeps through the x-axis as shown in Figure 1. The rule for handling the $\delta(\bar{y})dy$-integration, as noted in the

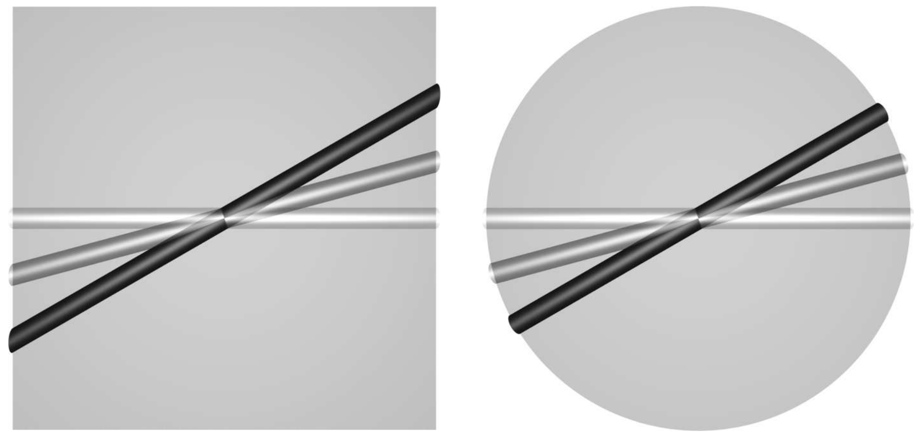

Figure 1: The figure on the left shows that the limits of integration is $x \in [-a, a]$ whereas the figure on the right shows the limits of integration as $x \in [-a\cos\omega t, a\cos\omega t]$.

Appendix[1], allows us to compute the following:

$$D_{11} = \frac{2}{3}\sigma a^3(3\cos^2\omega t - 1),$$
$$D_{12} = \sigma a^3 \sin(2\omega t),$$
$$D_{22} = \frac{2}{3}\sigma a^3(2 - 3\cos^2\omega t).$$

Applying this to (6), we get the well-known result of (7), completely independent of time and hence without the need to perform time-averaging.

Had we used fixed limits of integration, we would have found that the resulting dE/dt would have been *time-dependent* with an incorrect value $dE/dt = -10I^2\omega^6$ at time $t = 0$. (Furthermore, because of a $\cos\omega t$ factor in the denominator, the time-average would have been infinite as to be expected from an infinitely expanding rod as

[1]The rule in section 7.1, was aimed mainly at t being in the the neighbourhood of $t = 0$. However it will also apply for $-\pi/2 < \omega t < \pi/2$. Furthermore, without loss of generality, one can argue that the calculation is similar for $\pi/2 < \omega t < 3\pi/2$, but with the x's and y's interchanging roles.

$\omega t \to \pi/2$.) Thus, it is essential to bear in mind at all times that we are dealing with time-varying limits of integration. While it would have appeared simpler to perform the calculations in the co- moving frame as is commonly done using

$$D_{\alpha\beta} = \int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\int_{-a}^{a} \sigma\delta(\bar{z})\delta(\bar{y})(3x^{\alpha}(t)x^{\beta}(t) - \delta_{\alpha\beta}x^{\mu}(t)x^{\nu}(t)\delta_{\mu\nu})\, d\bar{x}d\bar{y}d\bar{z}$$

where $x(t) = \bar{x}\cos\omega t$ and $y(t) = \bar{x}\sin\omega t$, this would not serve our purposes in later work. The essential reason is that we will need to take multiple derivatives in the *inertial* spatial directions. Had we insisted on working in co-moving coordinates, we would have had to convert back to the inertial coordinates to take the spatial derivatives and then convert forward to the co-moving coordinates to perform the integration. As seen from the conversion from (15) to (16), this can be rather complicated. Hence, we confine our calculations to the inertial coordinates.[2]

4 The Extended Eddington Calculation

Having established the limits of integration, we now re-examine the Eddington calculation (9) as a second example. Eddington had a one-dimensional problem involving only time differentiation and hence was able to perform a simple line-integration. However, the computation of the time derivative of the Tolman integral (17) is considerably more complex. From the Eddington integral, (24) must be subtracted and the latter, which contains F^l as in (19), has space as well as time derivatives. Thus, one can no longer consider it a one-dimensional problem; one must take spatial derivatives in directions parallel to and perpendicular to the rod. Furthermore, the time-derivative of the Dirac-

[2]The only exception to this is the integration of the source point where $\bar{y} = 0$.

delta distribution must be accounted for [3] as well as the double-step functions which truncates the rod. Clearly, there is a vast requirement difference between Eddington's calculation and that of the Tolman integral derivative.

The calculation is based on (3) where η_{ij}=diag(1,-1,-1,-1) is the flat Minkowski metric and [4]

$$h_{ij}(x,y,z,t) = -4\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\int_{-a}^{a}\Big[\frac{T'_{ij}-\frac{1}{2}\eta_{ij}T'^{m}{}_{m}}{(1-v_r)r}\Big]_{\rm ret}d\bar{x}'d\bar{y}'d\bar{z}'$$

where $r \equiv R(x,y;\bar{x}',t) = \sqrt{(x-\bar{x}'\cos\omega t)^2+(y-\bar{x}'\sin\omega t)^2+z^2}$. Eddington used the present time expansion (11). This integration is performed in co-moving coordinates. The only place that $\bar{y}'$ comes into play is in the $\delta(\bar{y})$ and hence it is essentially zero after integration. For our purpose, we will not require the expansion beyond the 8^{th} power. The complete expression for h_{ij} is

$$\begin{aligned}h_{ij}(x,y,z,t) = -4\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\int_{-a}^{a}&\frac{T'_{ij}(\bar{x}',\bar{y}',\bar{z}',t)-\frac{1}{2}\eta_{ij}T'^{m}{}_{m}(\bar{x}',\bar{y}',\bar{z}',t)}{R(x,y,z;\bar{x}',t)}\\ &-\frac{d}{dt}\Big(T'_{ij}(\bar{x}',\bar{y}',\bar{z}',t)-\frac{1}{2}\eta_{ij}T'^{m}{}_{m}(\bar{x}',\bar{y}',\bar{z}',t)\Big)\\ +\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dt^n}\Big\{R(x,y,z;\bar{x}',t)^{n-1}&\Big(T'_{ij}(\bar{x}',\bar{y}',\bar{z}',t)-\frac{1}{2}\eta_{ij}T'^{m}{}_{m}(\bar{x}',\bar{y}',\bar{z}',t)\Big)\Big\}\,d\bar{x}'d\bar{y}'d\bar{z}'.\end{aligned}$$

It should be emphasized that

$$T'_{ij}(\bar{x}',\bar{y}',\bar{z}',t) = T_{ij}(x,y,z,t)\big|_{x=\bar{x}\cos\omega t-\bar{y}\sin\omega t,\,y=\bar{y}\cos\omega t+\bar{x}\sin\omega t,\,z=\bar{z}}$$

is an inertial entity but it is expressed in co-moving variables. Expressed in this form, there is no time-dependence in the Dirac-delta or the double-step functions and hence

[3]When Eddington did his calculation in 1922, the Dirac-delta distribution had not been established.

[4]It is possible to use $h_{ij} = -4\int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a}\Big[\frac{T'_{ij}-\frac{1}{2}\eta_{ij}T'^{m}{}_{m}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\Big]_{\rm ret}dx'dy'dz'$ and integrate in the inertial coordinate system but the integration would be extremely difficult; we choose to integrate with co-moving variables.

the time-derivatives in the expansion are straight-forward.

5 Maple Calculations

At this point, we supply the technical details that form the transition from the required time rate of change of the formal Tolman integral (the Eddington term minus (24)) to the Maple program that is used to compute it.

5.1 The Tolman Integrals

The original Eddington calculation,

$$\frac{d}{dt}\int \mathcal{T}_0^0\, dV = \frac{1}{2}\int T^{ij}\frac{\partial h_{ij}}{\partial t}\, dV$$

can easily be calculated using the following maple code

```
Eddington:= ... intxyz(1/2*TUPPER[i,j]*diff(hlower[i,j],t))
```

The summations have been omitted for clarity. *The full Maple code is available from the authors.*

There are a few minor difficulties in calculating (24). As shown in the Appendix, the first time-derivative is commutative with the time-dependent integral. Thus any terms having a single time derivative can be readily computed without having to be concerned about boundary terms. However, such boundary condition difficulties arising from higher derivatives do appear in the first and third terms of (24).

We use the definition for F^α in (19) to expand the second term of (24) to render it more manageable by Maple. It is

$$\frac{\partial}{\partial t}\int F^\alpha x^\alpha\, dV = -\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\int_{a}^{a}\frac{\partial}{\partial t}\Big(\frac{1}{2}T^{ij}h_{ij,\alpha} - T^{ij}h^{k\alpha}\eta_{ik}\Big)x^\alpha\, dxdydz$$

where $\eta^{i\alpha} = -\delta^{i\alpha}$ has been used. It is calculated using the following Maple codes:

```
intxyz(diff(1/2*TUPPER[i,j]*diff(hlower[i,j],xi[alpha]),t)*xi[alpha])
intxyz(diff(-TUPPER[i,j]*diff(HUPPER[k,alpha],xi[j])
                                        *eta[i,k],t)*xi[alpha])
```

The last terms are

$$\frac{\partial}{\partial t}\int T^{\alpha k} h_{\alpha k}\, dV = \int_a^a \int_{-\epsilon}^{\epsilon} \int_a^a \frac{\partial}{\partial t}\Big(T^{\alpha k} h_{\alpha k}\Big)\, dxdydz$$

with the equivalent code of

```
intxyz(diff(TUPPER[alpha,k]*hlower[alpha,k],t))
```

For the third term, we require (32) because of the occurrence of the double time-derivative.

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2}{\partial t^2}\int F^0 x^\alpha x^\alpha\, dV \\
&\qquad = -a\omega^2(f(a)+f(-a)) + \int_a^a \int_{-\epsilon}^{\epsilon} \int_a^a \frac{\partial^2}{\partial t^2}\Big(\frac{1}{4}T^{ij}h_{ij,0} - \frac{1}{2}T^{ij}h^{k0}_{,j}\eta_{ik}\Big)x^\alpha x^\alpha\, dxdydz
\end{aligned}$$

where the boundary terms are evaluated using the function,

$$f(x) = \int_a^a \int_{-\epsilon}^{\epsilon} \Big(\frac{1}{4}T^{ij}h_{ij,0} - \frac{1}{2}T^{ij}h^{k0}_{,j}\eta_{ik}\Big)x^\alpha x^\alpha\, dydz.$$

This is calculated using the regular procedure, `intxyz()`, and the "boundary" procedure, `intyz()`, as seen in the following code:

```
intxyz(diff(1/4*TUPPER[i,j]*diff(hlower[i,j],t)*(x^2+y^2+z^2)
  -1/2*TUPPER[i,j]*diff(HUPPER[k,0],xi[j])*eta[i,k]*(x^2+y^2+z^2),t$2))
intyz(1/4*TUPPER[i,j]*diff(hlower[i,j],t)*(x^2+y^2+z^2)
  -1/2*TUPPER[i,j]*diff(HUPPER[k,0],xi[j])*eta[i,k]*(x^2+y^2+z^2))
```

The `intyz()` lacks x-integration but evaluates the resulting functions at $x = -a$ and $x = a$. It also accounts for the $-a\omega^2$ factor as seen in the Appendix.

We find that *apart from the quadrupole-like term, the second spatial trace part of the Tolman integral has a vanishing time-rate of change after all the remaining terms are summed.* In the absence of that single term, one would conclude that the Tolman integral changes at the same rate as the first part, i.e. as the Eddington value (9), (7). A goal in future work would be to evaluate this term by a further iteration of the field equations. However, this appears to be a very difficult task and the choice of a different dynamical system as we discuss below might obviate the necessity of a second iteration.

It is to be noted that by breaking up the left hand side of (9) into a term that contains the determinant of the spatial part of the metric plus a term that carries the remainder which is of the form field/$\mathcal{T}_0^0$ product, we find that the time derivative of the remainder term is zero. Hence Eddington was actually calculating the time rate of change of what should be called the "material" energy, i.e. that which arises from the proper 3-volume integral.

5.2 The Angular-Momentum Integrals

At this point, we gather additional information from the consideration of angular momentum change of the source. Expanded, the expression for dL^{12}/dt (26) is

$$\begin{aligned}\frac{dL^{12}}{dt} &= \int xF^2 - yF^1\,dV \\ &= \int -\frac{x}{2}T^{ij}h_{ij,2} - xh^{i2}_{,j}T^{pj}\eta_{pi}\,dV - \int -\frac{y}{2}T^{ij}h_{ij,1} - yh^{i1}_{,j}T^{pj}\eta_{pi}\,dV\end{aligned}$$

The equivalent codes are

```
 intxyz(-x/2*TUPPER[i,j]*diff(hlower[i,j],y)
        -x*diff(HUPPER[i,2],xi[j])*TUPPER[p,j]*eta[p,i])
-intxyz(-y/2*TUPPER[i,j]*diff(hlower[i,j],x)
        -y*diff(HUPPER[i,1],xi[j])*TUPPER[p,j]*eta[p,i])
```

When this is evaluated, we find that

$$\frac{dL^{12}}{dt} = -\frac{32I^2\omega^5}{5} \tag{28}$$

Again, it is worth noting that a split of the right hand side of the above equation as we discussed in the case of the Eddington integral, leads to the analogous result: the calculation is the same as for the time rate of change of what we should rightfully call the "material" angular momentum.

Thus we see that $\frac{dE}{dt}=\omega\frac{dL}{dt}$ as expected for a rigid rod ($\frac{dI}{dt} = 0$) with kinetic energy $E=\frac{I\omega^2}{2}$ and angular momentum $L=I\omega$. We had also noted this result in our very early work that was based upon pseudotensorial fluxes. However, this relation arises from the classical kinetic energy and angular momentum connections with mass, moment of inertia and velocities. It is unclear whether or not this lowest order connection is appropriate to make the deductions regarding essentially higher order quantities and effects.

In addition, there are important caveats that we consider in the following section.

6 Summary and Concluding Discussions

We began by tracing the essential features that have made the issue of energy and its localization in general relativity so challenging. The work of Einstein and Eddington

were emphasized and as in earlier papers, we focused upon the problem of the Eddington spinning rod. An important attractive feature of this source of gravity waves is that to lowest order, it can be taken as rigid and hence the time variation derives solely from its variation in position. We noted that while Eddington considered the integral of $\mathcal{T}^{00}$ as the energy, some years later Tolman showed that for stationary systems, the energy is actually the integral of $\mathcal{T}_0^0 - \mathcal{T}_\alpha^\alpha$ where the second term accounts for contributions to the energy stemming from the presence of the gravitational field. Since the issue under consideration is that of the possible *gravitational* energy loss and since, if it exists, it is so incredibly minute in magnitude at least for the kind of sources that we are considering here, it is clear that this $\mathcal{T}_\alpha^\alpha$ term is of considerable interest.

In earlier work, we had noted that various researchers through the years had presented the change in the energy as calculated from an initial stationary state to a final (at least asymptotically) stationary state as the ultimate criterion for the existence and measure of an energy loss. We then argued that a necessary condition for an energy loss was therefore a realization of a variation of the Tolman integral during an intervening non-stationary period of motion. We had previously used the field information supplied by Eddington to compute the change in the Tolman integral at an instant but this was inadequate. In this paper, we presented a complete and detailed treatment of the fields and found that the combination of all of the field/T_{ik} product terms produced no net time variation of the spatial trace part of the Tolman integral. The final term in the equation giving the time rate of change of this trace part of the Tolman formula is precisely the trace of the quadrupole moment expression that is used untraced in linearized theory with pseudotensor to deduce the "quadrupole formula". However, in

the latter treatment, the expression is squared and hence its value is required only to lowest order. In the present treatment with the Tolman integral, it is not squared and the expression is required to sixth order in v/c. While one can impose the condition of rigidity to lowest order as Eddington had done, the field equations impose constraints on what is possible for the energy-momentum tensor at higher order and the properties of the material come into consideration. We faced this issue many years ago [28] for the problem of axially-symmetric free-fall using the pseudotensor and now we see the issue of structure appearing anew. The exploration of this issue in the present context presents a new and likely difficult challenge that we leave for future study.

As well, there are other issues to be considered:

1. *Is it sensible to be speaking of a localized energy in the first place?*

Various researchers through the years have denied the possibility of a localized energy in general relativity, primarily due to the non-tensorial aspect of the pseudotensor, but others such as Bondi [27] argued that energy must be localizable. Moreover, even the deniers tacitly admit a certain degree of localization in any case as they speak of a material system *losing* energy and that this loss is due to the gravity waves carrying away this energy. In other words, there is an energy to speak of *within* the material source as well as *within* the waves.

2. *What is the energy of the system during the particularly interesting gravity-wave emitting phase?*

The standard derivations of the Tolman integral expression for energy break down when the metric is time-dependent. Therefore the energy at that point is unknown. In fact it is well to consider what criteria one would employ to decide upon the correctness

of any energy expression candidate in the dynamic state. There very well may be an additional part beyond the Tolman integral to express the energy for a dynamical system that is a localized expression whose density involves the accelerations of the matter. If so, it would not always have a non-zero integral: the Tolman integral alone describes the total energy for a stationary source such as a rotating disk whose elements, by virtue of rotation, do accelerate. A useful goal for future research would be to determine if there is an additional part and if so, to deduce its value.

3. *Is the radiation-reaction type of calculation as solidly based in general relativity as it is in electromagnetism?*

We regard this as an issue because Maxwell theory is linear and general relativity is non-linear. The radiation reaction in Maxwell theory (at least time-averaged) is meaningfully related to the unambiguous Poynting vector flux of electromagnetic radiation energy over even a lengthy period of time, in part because of the linearity. However, in the analogous problem in general relativity, there is no assurance that such a calculation can yield a meaningful result for anything other than a possibly very short time period beyond stationarity during which any non-linearities have not had sufficient time to grow to a significant size. Thus, while Eddington [9] argued that $v^2 \gg m/r$ for his spinning rod was a sufficient condition for a linearized treatment of the energy problem, the validity of linearization might be more severely restricted than he had believed. This has further ramifications, as we now discuss.

One might have argued that given the large reservoir of kinetic energy in a rapidly rotating Eddington rod as compared to the relatively small amount of gravitational contribution to the energy, and given the computed non-zero loss rate of the former (at

least within the context of the assumed validity of linearization), it is inevitable that over a long period of time, the loss would accumulate to a large net energy loss value that could never be compensated by any gravitational component. However, question 3 above is relevant here. It is possible that the kinetic energy loss as computed traditionally holds only for a relatively short period beyond stationarity and that during this phase, there still remains an equalizing *dynamical* gravitational component to render the localized energy conserved within the source. Moreover, as the non-linearities grow, it is possible that the kinetic energy loss component levels out and the vast kinetic energy reservoir can never really be tapped. Thus, an experimental test of the localization hypothesis would be one of actually observing the equivalent of a rapidly rotating *strongly bound* (i.e. with $v^2 \gg m/r$) system such as an Eddington rod. The rate of change in period would have to decrease with the system approaching a state of uniform rotation as in classical physics for the viability of the hypothesis. On the other hand, if the decay should continue unabated, ultimately draining away the vast reservoir of kinetic energy, then the traditional view of an energy loss carried by the gravitational waves would be supported. Unfortunately, such strongly bound systems of sufficient strength to yield detectable results would be very hard to come by and the far more complex *gravitationally* bound system of a rotating binary such as PSR 1913+16 would not furnish such a test since in such a case, the store of gravitational energy is necessarily comparable to that of the kinetic energy store, i.e. v^2 of the order of m/r.

We are presently exploring a problem that will hopefully by-pass the difficulties inherent in the Eddington source. It is based upon an axially symmetric source that evolves from a *static* configuration, evolves into a dynamic phase with an imposed variation in

its energy-momentum tensor and then returns after a short time interval to a final rest configuration at lowest order. The goal is to integrate the change in the Tolman integral over the brief history of evolution to determine whether or not the active gravitational mass at the end equals the value at the beginning. By this approach, one avoids the uncertainty associated with the potential build-up of non-linearities. Moreover, by completing the dynamic phase, the change or constancy in the Tolman integral would answer the energy-loss question in a definitive manner.

7 Appendix

7.1 Dirac-Delta Integration

The integral of

$$\int_{-\epsilon}^{\epsilon} f(x,y,t)\delta(y\cos\omega t - x\sin\omega t)\,dy$$

is evaluated using the well-known identity $\delta(ay-b) = \frac{1}{|a|}\delta(y-\frac{b}{a})$. We assume that t is in the neighhbourhood of $t=0$ and so for some $\epsilon > 0$, the limits of integrations will completely cover the Dirac-delta distribution. Factoring the coefficient out and evaluating the integral results in

$$\int_{-\epsilon}^{\epsilon} f(x,y,t)\frac{1}{|\cos\omega t|}\delta\Big(y - \frac{x\sin\omega t}{\cos\omega t}\Big)\,dy = \frac{f(x, x\tan\omega t, t)}{\cos\omega t}$$

where we have assumed that $|\cos\omega t| \equiv \cos\omega t$ in the neighbourhood of $t=0$. Thus the rule for integrating $\delta(-x\sin\omega t + y\cos\omega t)$ WRT y is to substitute $y = x\tan\omega t$ and divide the entire quantity by $\cos\omega t$.

7.2 Single Time-Derivative and Integral Commutation

It is very impractical to integrate before setting time $t = 0$; setting $t = 0$ prematurely could remove a large number of terms that are required for integration. The identity,

$$\Big\{\frac{\partial}{\partial t}\int_{-a\cos\omega t}^{a\cos\omega t} f(x,t)dx\Big\}\Big|_{t=0} = \int_{-a}^{a}\Big\{\frac{\partial f(x',t)}{\partial t}\Big\}\Big|_{t=0} dx' \tag{29}$$

for an arbitrary function $f(x,t)$ is extremely useful. It allows one to bring the time-derivative through the time-dependent integral. The proof of this identity is as follows. [5] Define a function $F(x,t)$ as

$$F(x,t) = \int_0^x f(x',t)dx' \tag{30}$$

with the condition that $\partial^2 F(x,t)/\partial x\partial t = \partial^2 F(x,t)/\partial t\partial x$. Using this definition in the integral on the LHS of (29), we find that

$$\begin{aligned}\int_{-a\cos\omega t}^{a\cos\omega t} f(x,t)dx &= F(x,t)\big|_{x=a\cos\omega t} - F(x,t)\big|_{x=-a\cos\omega t}\\ &= F(a\cos\omega t,t) - F(-a\cos\omega t,t).\end{aligned}$$

When we apply the time-derivative, the chain rule is used on the $x = \pm a\cos\omega t$ terms and the equation becomes

$$\begin{aligned}\frac{\partial}{\partial t}\int_{-a\cos\omega t}^{a\cos\omega t} f(x,t)dx =&\Big(\frac{\partial F(x,t)}{\partial x}\frac{d}{dt}(a\cos\omega t) + \frac{\partial F(x,t)}{\partial t}\Big)\Big|_{x=a\cos\omega t}\\ &- \Big(\frac{\partial F(x,t)}{\partial x}\frac{d}{dt}(-a\cos\omega t) + \frac{\partial F(x,t)}{\partial t}\Big)\Big|_{x=-a\cos\omega t}\\ =&- a\omega\sin\omega t\,(f(a\cos\omega t,t) + f(-a\cos\omega t,t))\\ &+ \Big(\frac{\partial F(x,t)}{\partial t}\Big)\Big|_{x=a\cos\omega t} - \Big(\frac{\partial F(x,t)}{\partial t}\Big)\Big|_{x=-a\cos\omega t}\end{aligned} \tag{31}$$

[5] Of course this definition of $F(x,t)$ is not unique. Also, note that we could have defined $F(x,t) = \int_0^x f(x',t)dx'$ but we cannot be sure that there will be no singularities in the domain $[0,x]$.

The link, $x = a\cos\omega t$, is applied at the very last step–*outside the parenthesis*. Hence, there is no need to be concerned about time-varying integral limits when we do take the time-derivative of (30). Now, once we set $t = 0$, we find that the $\sin\omega t$ terms go to zero simplifying the equation,

$$\begin{aligned}\Big\{\frac{\partial}{\partial t}\int_{-a\cos\omega t}^{a\cos\omega t} f(x,t)dx\Big\}\Big|_{t=0} =& \Big(0 + \frac{\partial F(x,t)}{\partial t}\Big)\Big|_{x=a,t=0} \\ &- \Big(0 + \frac{\partial F(x,t)}{\partial t}\Big)\Big|_{x=-a,t=0}.\end{aligned}$$

To show that this is indeed the RHS of (29), we refer back to the definition of $F(x,t)$ in (30). The expression,

$$\Big(\frac{\partial F(x,t)}{\partial t}\Big)\Big|_{x=a,t=0} - \Big(\frac{\partial F(x,t)}{\partial t}\Big)\Big|_{x=-a,t=0} = \Big(\frac{\partial}{\partial t}\int_{-a}^{a} f(x',t)dx'\Big)\Big|_{t=0}$$

is equivalent to the RHS of (29) because we can now move the time-derivative through the integral.

7.3 Double Time-Derivative and Integral Commutation

With a similar approach we can find an expression for the second time-derivative as we now demonstrate:

$$\begin{aligned}&\Big\{\frac{\partial^2}{\partial t^2}\int_{-a\cos\omega t}^{a\cos\omega t} f(x,t)\,dx\Big\}\Big|_{t=0} \\ &\qquad = -a\omega^2(f(a,0) + f(-a,0)) + \int_{-a}^{a}\Big\{\frac{\partial^2 f(x',t)}{\partial t^2}\Big\}\Big|_{t=0} dx'. \quad (32)\end{aligned}$$

To prove this, we first define an intermediary function,

$$G(x,t) = \frac{\partial F(x,t)}{\partial t}. \quad (33)$$

When we apply another time-derivative to (31) with the help of (33), it becomes

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\int_{-a\cos\omega t}^{a\cos\omega t} f(x,t)\,dx =& \frac{\partial}{\partial t}\Big(-a\omega\sin\omega t\,(f(a\cos\omega t,t)+f(-a\cos\omega t,t))\Big)\\ &+\frac{\partial}{\partial t}\Big(G(x,t)\big|_{x=a\cos\omega t}-G(x,t)\big|_{x=-a\cos\omega t}\Big).\end{aligned}$$

We will not need to expand the first two terms completely because when we set $t=0$, many parts of

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\int_{-a\cos\omega t}^{a\cos\omega t} f(x,t)dx =& -a\omega^2\cos\omega t\Big(f(a\cos\omega t,t)+f(-a\cos\omega t,t)\Big)\\ &-a\omega\sin\omega t\frac{\partial}{\partial t}\Big(f(a\cos\omega t,t)+f(-a\cos\omega t,t)\Big)\\ &+\Big(\frac{\partial G(x,t)}{\partial x}\frac{d}{dt}(a\cos\omega t)+\frac{\partial G(x,t)}{\partial t}\Big)\Big|_{x=a\cos\omega t}\\ &-\Big(\frac{\partial G(x,t)}{\partial x}\frac{d}{dt}(-a\cos\omega t)+\frac{\partial G(x,t)}{\partial t}\Big)\Big|_{x=-a\cos\omega t}\end{aligned}$$

will cancel. With $t=0$, the expression simplifies to

$$\begin{aligned}\Big\{\frac{\partial^2}{\partial t^2}\int_{-a\cos\omega t}^{a\cos\omega t} f(x,t)\,dx\Big\}\Big|_{t=0} =& -a\omega^2\Big(f(a,0)+f(-a,0)\Big)\\ &+\Big(\frac{\partial G(x,t)}{\partial t}\Big)\Big|_{x=a,t=0}-\Big(\frac{\partial G(x,t)}{\partial t}\Big)\Big|_{x=-a,t=0}.\end{aligned}$$

Based on the definition for $G(x,t)$ in (33), this is

$$\begin{aligned}\Big\{\frac{\partial^2}{\partial t^2}\int_{-a\cos\omega t}^{a\cos\omega t} f(x,t)\,dx\Big\}\Big|_{t=0} =& -a\omega^2\Big(f(a,0)+f(-a,0)\Big)\\ &+\Big(\frac{\partial^2 F(x,t)}{\partial t^2}\Big)\Big|_{x=a,t=0}-\Big(\frac{\partial^2 F(x,t)}{\partial t^2}\Big)\Big|_{x=-a,t=0}.\end{aligned}$$

And finally, using (30), we can see that this

$$\begin{aligned}&-a\omega^2\Big(f(a,0)+f(-a,0)\Big)+\Big(\frac{\partial^2 F(x,t)}{\partial t^2}\Big)\Big|_{x=a,t=0}-\Big(\frac{\partial^2 F(x,t)}{\partial t^2}\Big)\Big|_{x=-a,t=0}\\ &\qquad= -a\omega^2\Big(f(a,0)+f(-a,0)\Big)+\Big\{\frac{\partial^2}{\partial t^2}\int_{-a}^{a} f(x',t)dx'\Big\}\Big|_{t=0}\end{aligned}$$

is equivalent to the RHS of (32).

7.4 Single Time-Derivative of Step-Function

Another proof of section 7.2 can be done using step-functions. The basic reason why there is time-dependence in the integral limits of (29) is because of the step-functions in the integrand. An equivalent statement to (29) is

$$\begin{aligned}\Big\{\frac{\partial}{\partial t}\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}(H(a-\bar{x})+H(a+\bar{x})-1)g(x,y,t)\delta(\bar{y})\,dydx\Big\}_{|t=0}\\ =\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}\Big\{\frac{\partial}{\partial t}\Big(g(x,y,t)\delta(\bar{y})\Big)\Big\}_{|t=0}dydx \quad (34)\end{aligned}$$

with the relationship between (29) and (34) being $f(x,t) = \int_{-\epsilon}^{\epsilon} g(x,y,t)\delta(\bar{y})\,dy = g(x, x\tan\omega t, t)/\cos\omega t$. We can move the time derivative through the integral of (34), yielding

$$\begin{aligned}\frac{\partial}{\partial t}\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}(H(a-\bar{x})+H(a+\bar{x})-1)g(x,y,t)\delta(\bar{y})\,dydx\\ =\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}(-\omega\bar{y}\delta(\bar{x}-a)+\omega\bar{y}\delta(\bar{x}+a))g(x,y,t)\delta(\bar{y})\\ +(H(a-\bar{x})+H(a+\bar{x})-1)\frac{\partial}{\partial t}\Big(g(x,y,t)\delta(\bar{y})\Big)\,dydx. \quad (35)\end{aligned}$$

Having now dispensed with the derivative apart from in the very last term, we may set $t=0$. This implies that $\bar{x}=x$, $\bar{y}\to 0$, and $H(a\pm\bar{x})\equiv 1$. Thus,

$$\begin{aligned}\Big\{\frac{\partial}{\partial t}\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}(H(a-\bar{x})+H(a+\bar{x})-1)g(x,y,t)\delta(\bar{y})\,dydx\Big\}_{|t=0}\\ =\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}(-\omega y\delta(y)\delta(x-a)+\omega y\delta(y)\delta(x+a))g(x,y,0)\\ +(1)\frac{\partial}{\partial t}\Big(g(x,y,t)\delta(\bar{y})\Big)_{|t=0}dydx.\end{aligned}$$

Using the fact that $y\delta(y)\equiv 0$, the first two terms vanish, leading us to (34).

7.5 Double Time-Derivative of Step-Function

The equivalent statement to (32) using step functions is

$$\left\{\frac{\partial^2}{\partial t^2}\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}(H(a-\bar{x})+H(a+\bar{x})-1)g(x,y,t)\delta(\bar{y})\,dydx\right\}_{\big|t=0}$$
$$= -a\omega^2\Big(g(a,0,0)+g(-a,0,0)\Big)+\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}\left\{\frac{\partial^2}{\partial t^2}\Big(g(x,y,t)\delta(\bar{y})\Big)\right\}_{\big|t=0}dydx. \quad (36)$$

To prove this, we first apply another time-derivative to (35) to get

$$\frac{\partial^2}{\partial t^2}\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}(H(a-\bar{x})+H(a+\bar{x})-1)g(x,y,t)\delta(\bar{y})\,dydx$$
$$=\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}\Big(\omega^2\bar{x}\delta(\bar{x}-a)+\omega^2\bar{y}^2\frac{\delta(\bar{x}-a)}{(a-\bar{x})}-\omega^2\bar{x}\delta(\bar{x}+a)-\omega^2\bar{y}^2\frac{\delta(\bar{x}+a)}{-(\bar{x}+a)}\Big)g(x,y,t)\delta(\bar{y})$$
$$+2\Big(-\omega\delta(\bar{x}-a)\bar{y}+\omega\delta(\bar{x}+a)\bar{y}\Big)\Big(\delta(\bar{y})\frac{\partial g(x,y,t)}{\partial t}+\frac{\omega\bar{x}}{\bar{y}}\delta(\bar{y})g(x,y,t)\Big)$$
$$+(H(a-\bar{x})+H(a+\bar{x})-1)\frac{\partial^2}{\partial t^2}\Big(g(x,y,t)\delta(\bar{y})\Big)\,dydx.$$

Now, set $t=0$ and we get

$$\left\{\frac{\partial^2}{\partial t^2}\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}(H(a-\bar{x})+H(a+\bar{x})-1)g(x,y,t)\delta(\bar{y})\,dydx\right\}_{\big|t=0}$$
$$=\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}\Big(\omega^2x\delta(x-a)-\omega^2x\delta(x+a)\Big)g(x,y,0)\delta(y)$$
$$+2\Big(-\omega\delta(x-a)y+\omega\delta(x+a)y\Big)\frac{\omega x}{y}\delta(y)g(x,y,0)$$
$$+(1)\left\{\frac{\partial^2}{\partial t^2}\Big(g(x,y,t)\delta(\bar{y})\Big)\right\}_{\big|t=0}dydx$$

The first four terms of the RHS collapse to $-a\omega^2(g(a,0,0)+g(-a,0,0))$ and hence the whole expression,

$$-a\omega^2(g(a,0,0)+g(-a,0,0))+\int_{-a}^{a}\int_{-\epsilon}^{\epsilon}\left\{\frac{\partial^2}{\partial t^2}\Big(g(x,y,t)\delta(\bar{y})\Big)\right\}_{\big|t=0}dydx,$$

is equivalent of the RHS of (36)

Acknowledgment: This work was supported by grants from the Natural Sciences and Engineering Research Council of Canada and the University of Victoria Faculty Research Fund. We thank M. Santoprete for his kind assistance.

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